Effective Elastic Properties of Damaged Isotropic Solids

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In continuum damage mechanics, damaged solids have been represented by the effective elastic stiffness into which local damage is smoothly smeared. Similarly, damaged solids may be represented in terms of effective elastic compliances. By virtue of the effective elastic compliance representation, it may become easier to derive the effective engineering constants of damaged solids from the effective elastic compliances, all in closed form. Thus, in this paper, by using a continuum modeling approach based on both the principle of strain energy equivalence and the equivalent elliptical micro-crack representation of local damage, the effective elastic compliance and effective engineering constants are derived in terms of the undamaged (virgin) elastic properties and a scalar damage variable for both damaged two- and three-dimensional isotropic solids.

Key Words: Continuum Damage Mechanics, Effective Compliance, Effective Engineering Constants, Damaged Solid, Isotropic Solid

1. Introduction

Since the ability to accurately predict the remaining operational life of a mechanical system is an essential element of a condition-based maintenance approach (Hansen *et al.*, 1995), numerous approaches to the characterization and prediction of damage birth, growth, and evolution have appeared in the literature associated with various fields, including physics, applied mathematics, material sciences and engineering, fracture mechanics, and damage mechanics (see Hu *et al.*, 1988). A material failure process is often assumed to involve general degradation of elasticity properties due to the highly localized nucleation and growth of micro-defects (*i. e.*, micro-cracks and micro-voids) and their ultimate coalescence into macrodefects. The process and result of these irreversible, energy dissipating, microstructural rearrangements is often called damage. Damage variables have been introduced to reflect average material degradation on a macro-mechanics scale normally associated with the classical theory of continuum mechanics. This innovative concept of continuum damage mechanics (CDM) was originally proposed by Kachanov (1958) and has been extended to solve important practical engineering problems. Extensive treatments of CDM can be found in the book by Krajcinovic (1996).

In CDM, there has been extensive research focusing on two major subjects: constitutive laws (i. e., stress-strain relationships) of damaged solids, and damage evolution equations. The constitutive laws and damage evolution equations are closely related to each other, as well as to the definition of damage variables in nature. Current theories of CDM may be classified into four categories on the basis of (a) the type of constitutive law of the damaged solid; and (b) the scalar, vector, or tensor nature of damage variables and evolution equations (Lee *et al.*, 1997). Most theories of CDM have been developed for initially (virgin) *isotropic* solids. To the author's best knowledge, the damage theory recently developed by Lee *et al.* (1997) is the first that permits anisotropic behavior of the damaged isotropic material, while using a scalar damage variable.

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Very recently, Lee (1997a) also has developed a theory of CDM for initially (virgin) anisotropic solids, hopefully applicable to composite materials which show in general anisotropic behavior.

As recognized by Lemaitre and Chaboche (1978), interpreting the damage as a change in the elastic moduli rather than a change in the stress transmitting surface area is a convenient and correct approach. Since damage degrades the material properties, material response modeling must address the formulation of constitutive laws, a task which may be approached using micro-mechanical and/or phenomenological approaches. A brief review of such approaches is given in the following.

The homogenization of the meso-structural field of defects within a representative volume element (RVE) into a macro-field of the effective continuum corresponds to micro-mechanical modeling. One of the micro-mechanical approaches appearing in the literature is the so-called self-consistent method attributable to Hershey (1954). This method was originally proposed for aggregates of crystals, and has emerged as one of the simplest effective continuum modeling methods. In this method, it is usually assumed that the averaged strain (or strain energy) within a RVE is approximated in terms of the as-yet-unknown macroscopic elastic moduli (Budiansky and O'Connell, 1976; Hoeing, 1979; Horri and Nemat-Nasser, 1983; Laws and Dvorak, 1987). Despite clarity and a well-defined relationship with physical phenomena, the self-consistent method based on the stochastic defects within an RVE may require accurate prediction and/or measurement of the micro-crack density parameters, which may be impractical or impossible, especially during the phases of crack nucleation and growth. There is an alternate continuum method introduced in Lee's damage theory (hereafter referred to as LDT) (Lee *et al.,* 1997 and Lee, 1997). The continuum method in LDT is somewhat similar to the self-consistent method, in that both methods consider the strain energy of damaged solids in the process of homogenization. However, in LDT, the effective continuum is derived for a material volume cell (MVC) containing a single fictitious elliptical micro-crack that represents the local damage. Thus, the continuum method in LDT differs from the self-consistent method both in spirit and in detail.

In contrast to micro-mechanical models, phenomenological models do not consider the micro-details of material response, but describe damage indirectly by introducing internal (or hidden) variables. This has caused some confusion and spawned more extensive, substantially different models of the same phenomena. Since the selection of the damage variable is perhaps the most important step, irreversible thermodynamics has been used to provide a scientific basis for theories of CDM. In the literature, diverse definitions of damage variables have been introduced. They are scalar variables (Kachanov, 1958), vector variables (Davison and Stevens, 1973), second order tensors (Vakulenko and Kachanov, 1971), and fourth order tensors (Chaboche, 1979). Recently, Lee *et al.* (1997) introduced a new scalar damage variable D on the basis of the crack energy in fracture mechanics, and it was consistently used to develop a damage evolution equation and effective continuum stiffness of damaged solids.

To formulate the constitutive laws of damaged solids, the strain equivalence principle (Lemaitre, 1985), stress equivalence principle (Simo and Ju, 1987), and strain energy equivalence principle (SEEP) (Lee *et al.,* 1997) were introduced in the literature. In SEEP, a material volume cell (MVC) of the damaged solid containing a local damage and its equivalent continuum model (ECM) are assumed to contain equal strain energy when they are subject to identical global displacements on their boundaries. The SEEP has the apparent advantage over the first two principles in that the complete constitutive law can be obtained by simply replacing the undamaged (virgin) elastic stiffness with the effective elastic stiffness derived on the basis of SEEP, without redefining or changing the nominal stresses or strains appearing in the original constitutive law. This approach seems consistent with the physical interpretation of elastic moduli degradation due to reduction in the effective stress-transmitting area (Lubarda et al. 1994). Though Chow and Wang (1987) used a similar terminology "elastic energy equivalence" in their CDM, it differs from SEEP because they used the "fictitious" elastic energy formulated on the basis of the strain equivalence principle.

As mentioned above, Lee *et al.* (1997) developed a new continuum damage theory (LDT) on the basis of both the principle of strain energy equivalence and the concept of equivalent fictitious elliptical micro-crack representation of a local damage. In LDT, damaged solids are represented in terms of effective elastic stiffnesses into which local damage is smoothly smeared. In parallel to the effective elastic stiffness representation, damaged solids can be represented in terms of effective elastic compliance. By virtue of the effective elastic compliance representation, it may become easier to derive the effective engineering constants (*i. e.*, effective elastic moduli and Poisson's ratios) in closed forms from effective elastic compliance, and to identify local damage states by simply observing the changes in effective engineering constants due to damage evolution.

Thus, in this paper, the effective elastic compliances and the effective engineering constants are derived in closed form for both two- and three-dimensional isotropic solids which show anisotropic material behaviors after damage.

2. Effective Elastic Compliances

In deriving the effective elastic compliances of damaged isotropic solids, the continuum modeling procedure of LDT will be followed in this paper. First, isolate a MVC from an initially virgin isotropic solid so that it contains a single micro-crack (*i.e.*, a local damage). Then the damaged MVC will be modeled as an ECM by determining its effective elastic compliance on the basis of SEEP. As shown in Fig. 1, the damaged MVC has the characteristic dimension, *i. e.*, radius R. The stress-strain relation for an undamaged isotropic material can be written in terms of undamaged (virgin) isotropic elastic compliances S_{ij} as

$$\{\varepsilon\}=[S_{ij}]\{\sigma\} \quad \text{or} \quad \varepsilon=S\sigma \tag{1}$$

and, for the ECM, it can be written in terms of effective (damaged) elastic compliances $\bar{S}_{ij}$ as

$$\{\varepsilon\}=[\bar{S}_{ij}]\{\sigma\} \quad \text{or} \quad \varepsilon=\bar{S}\sigma \tag{2}$$

where contracted notation is used for both stresses and strains, *i. e.*, $\sigma_1=\sigma_{11}$, $\sigma_2=\sigma_{22}$, $\sigma_4=\sigma_{23}$, $\sigma_5=\sigma_{31}$, $\sigma_6=\sigma_{12}$, $\varepsilon_1=\varepsilon_{11}$, $\varepsilon_2=\varepsilon_{22}$, $\varepsilon_3=\varepsilon_{33}$, $\varepsilon_4=2\varepsilon_{23}$, $\varepsilon_5=2\varepsilon_{31}$, and $\varepsilon_6=2\varepsilon_{12}$. In Eqs. (1) and (2), i, $j=1, 2, 6$ for plane problems while $i, j=1, 2, \cdots, 6$ for three-dimensional cases.

Since the macro-behavior represented by ECM should be the same as that of damaged MVC, we may determine the effective elastic compliances such that they contain equal strain energy when they are subject to identical stress distribution, σ, at the same radial distance R. Thus, SEEP may provide the effective elastic compliances of ECM by equating the strain energy V_d contained in damaged MVC to the strain energy V_{eq} in ECM. That is,

$$V_{eq}(\bar{S};\sigma)=V_d(S,\ D;\sigma) \tag{3}$$

where D represents a scalar damage variable which will be defined in the following.

The strain energy V_{eq} that can be stored in a homogenized ECM may be written in terms of effective elastic compliances $\bar{S}_{ij}$ as

$$V_{eq}=\frac{1}{2}\pi R^2\{\sigma\}^T[\bar{S}]\{\sigma\} \qquad \text{for 2-D problems} \tag{4a}$$

$$V_{eq}=\frac{1}{2}\left(\frac{4}{3}\pi R^3\right)\{\sigma\}^T[\bar{S}]\{\sigma\} \quad \text{for 3-D problems} \tag{4b}$$

In fracture mechanics, the strain energy that can be stored in damaged MVC is obtained in general form as (Sih and Liebowitz, 1968)

$$V_d=V_o+V_c \tag{5}$$

where V_o is the strain energy for undamaged MVC given as

$$V_o=\frac{1}{2}\pi R^2\{\sigma\}^T[S]\{\sigma\} \qquad \text{for 2-D problems} \tag{6a}$$

$$V_o=\frac{1}{2}\left(\frac{4}{3}\pi R^3\right)\{\sigma\}^T[S]\{\sigma\} \quad \text{for 3-D problems} \tag{6b}$$

and V_c is the strain energy released in forming a crack, often called the crack energy. The crack energies V_c for both the elliptical through-crack (*i. e.*, elliptic hole) in a two-dimensional elastic solid and the elliptical plane-crack embedded in a three-dimensional elastic solid are available from Sih and Liebowitz (1968).

2.1 Two-dimensional damages: plane problems

Consider a two-dimensional MVC of isotropic solid, which contains an elliptical through-crack under bi-axial normal stresses (σ_1 and σ_2) and in-plane shear stress (σ_6) at the radial distance R as shown in Fig. 1(a). The center of the elliptical crack is located at the origin of the crack coordinates (1-2), with the major axis (length $2a$) and the minor axis (length $2b$) aligned with the coordinates "1" and "2", respectively.

For generalized plane stress state, undamaged (virgin) isotropic elastic compliances are given in terms of engineering constants (*i. e.,* Young's modulus E, shear modulus G, and Poisson's ratio ν) as follows:

$$S_{11}=S_{22}=\frac{1}{E},\quad S_{66}=\frac{1}{G}=\frac{2(2+\nu)}{E},$$
$$S_{12}=-\frac{\nu}{E},\quad S_{16}=S_{26}=0 \tag{7}$$

The elastic compliances for plane strain can be obtained by simply replacing E and ν in Eq. (7) with $E/(1-\nu^2)$ and $\nu/(1-\nu)$, respectively. Thus, the formulations derived in the following for the generalized plane stress state can be converted to those for the plane strain state exactly in the same way.

The crack energy due to an elliptical through-crack was derived by Sih and Liebowitz (1968), and it can be rewritten in the form:

$$V_c=\frac{1}{2}\pi a^2\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{Bmatrix}^{T}\begin{bmatrix}S_{11}\bar{e}_{11} & S_{12}\bar{e}_{12} & 0\\ S_{12}\bar{e}_{12} & S_{22}\bar{e}_{22} & 0\\ 0 & 0 & S_{66}\bar{e}_{66}\end{bmatrix}\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{Bmatrix} \tag{8}$$

where the parameters $\bar{e}_{ij}(\nu,\ k')$ are defined as

$$\bar{e}_{11}=2k'^2+k',\quad \bar{e}_{22}=2+k',\quad \bar{e}_{12}=\frac{k'}{\nu},$$
$$\bar{e}_{66}=\frac{(1+k')^2}{1+\nu} \tag{9}$$

In Eq. (9), $k'=b/a$ is the aspect ratio of the elliptical crack with values $0\le k'\le 1$. The parameters $\bar{e}_{ij}(\nu,\ k')$ may reflect the current state of a local damage in a solid. The over-bar has been used in the preceding equations in order to distinguish the parameters $\bar{e}_{ij}$ from the parameters e_{ij} defined in Lee *et al.* (1997) for the effective elastic stiffness.

Substituting Eqs. (6a) and (8) into Eq. (5) yields

$$V_d=\frac{1}{2}\pi R^2\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{Bmatrix}^{T}\begin{bmatrix}S_{11}(1+\bar{e}_{11}D) & S_{11}(1+\bar{e}_{12}D) & 0\\ S_{12}(1+\bar{e}_{12}D) & S_{22}(1+\bar{e}_{22}D) & 0\\ 0 & 0 & S_{66}(1+\bar{e}_{66}D)\end{bmatrix}\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{Bmatrix} \tag{10}$$

where D is a scalar variable defined as

$$D=\left(\frac{a}{R}\right)^2 \tag{11}$$

The scalar variable D is a new definition of damage variable introduced in LDT (Lee *et al.,* 1997). This new damage variable may be interpreted as the ratio of the effective damaged area (πa^2) to the initially virgin area of MVC (πR^2), which differs from the classical damage variable defined as the surface density of intersection of micro-cracks and micro-voids with any plane in the body (Lemaitre, 1992).

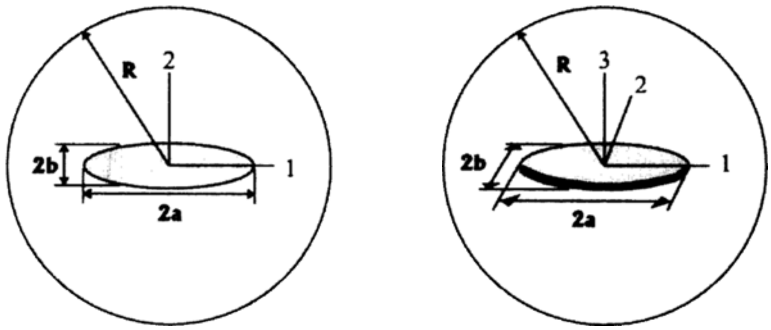
On the basis of SEEP expressed by Eq. (3), the effective elastic compliances are now obtained as follows:

$$\bar{S}_{11}=\frac{1}{E}(1+\bar{e}_{11}D),\quad \bar{S}_{22}=\frac{1}{E}(1+\bar{e}_{22}D)$$
$$\bar{S}_{12}=-\frac{\nu}{E}(1+\bar{e}_{12}D),\quad \bar{S}_{66}=\frac{1}{G}(1+\bar{e}_{66}D) \tag{12}$$
$$\bar{S}_{16}=\bar{S}_{26}=0$$

It is obvious from Eq. (12) that damaged MVC of two-dimensional isotropic solid in general shows orthotropic macro-behavior. For the case of circular through-cracks (*i. e.,* circular hole) ($k'=1$), it can be shown that $\bar{S}_{66}=2(\bar{S}_{11}-\bar{S}_{12})$, and thus it behaves isotropically as expected.

2.2 Three-dimensional damages

Consider a three-dimensional MVC of isotropic solid, with an embedded elliptical plane-crack as shown in Fig. 1(b). The origin of crack coordinates (1-2-3) is located at the center of the



(a) Elliptical through-crack (*i.e.*, elliptic hole) in two-dimensional solid

(b) Elliptical plane-crack embedded in three-dimensional solid.

Fig. 1 Illustration of the damaged material volume cells (MVC) containing single elliptical cracks considered in the present study.

ellipse. The coordinate directions "1" and "2" are aligned with the major axis (length $2a$) and the minor axis (length $2b$) of the crack, respectively, while the coordinate "3" is normal to the plane of the crack.

For three-dimensional isotropic solids, undamaged (virgin) elastic compliances are given in terms of engineering constants as follows:

$$S_{11}=S_{22}=S_{33}=\frac{1}{E},\ S_{44}=S_{55}=S_{66}=\frac{1}{G} \tag{13}$$

$$S_{12}=S_{13}=S_{23}=-\frac{\nu}{E},\quad \textit{other}\ S_{ij}=0$$

The crack energy due to an elliptical plane-crack embedded in a three-dimensional solid under general loading was also derived by Kassir and Sih (1967), and it can be rewritten in the form of

$$V_c=\frac{1}{2}\left(\frac{4\pi a^3}{3}\right)\left(\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{Bmatrix}^T\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & S_{33}\bar{e}_{33}\end{bmatrix}\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{Bmatrix}+\begin{Bmatrix}\sigma_4\\ \sigma_5\\ \sigma_6\end{Bmatrix}^T\begin{bmatrix}S_{44}\bar{e}_{44} & 0 & 0\\ 0 & S_{55}\bar{e}_{55} & 0\\ 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}\sigma_4\\ \sigma_5\\ \sigma_6\end{Bmatrix}\right) \tag{14}$$

where

$$\begin{aligned}\bar{e}_{33}&=\frac{2(1-\nu^2)k'^2}{E(k)}\\ \bar{e}_{44}&=\frac{(1-\nu)k'^2k^2}{(k^2+\nu k'^2)E(k)-\nu k'^2K(k)}\\ \bar{e}_{55}&=\frac{(1-\nu)k'^2k^2}{(k^2-\nu)E(k)+\nu k'^2K(k)}\\ k^2&=1-k'^2\end{aligned} \tag{15}$$

In the preceding equations, $K(k)$ is the complete elliptic integral of the first kind, and $E(k)$ of the second kind. For the penny-shaped circular cracks ($k'=1$), $\bar{e}_{ij}$ are simplified to

$$\bar{e}_{33}=\frac{4}{\pi}(1-\nu^2),\ \bar{e}_{44}=\bar{e}_{55}=\frac{4}{\pi}\left(\frac{1-\nu}{2-\nu}\right) \tag{16}$$

Substituting Eqs. (6b) and (14) into Eq. (5) yields the strain energy stored in a damaged MVC as follows:

$$V_d=\frac{1}{2}\left(\frac{4\pi R^3}{3}\right)\left(\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{Bmatrix}^T\begin{bmatrix}S_{11} & S_{12} & S_{13}\\ S_{12} & S_{22} & S_{23}\\ S_{13} & S_{23} & S_{33}(1+\bar{e}_{33}D)\end{bmatrix}\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{Bmatrix}+\begin{Bmatrix}\sigma_4\\ \sigma_5\\ \sigma_6\end{Bmatrix}^T\begin{bmatrix}S_{44}(1+\bar{e}_{44}D) & 0 & 0\\ 0 & S_{55}(1+\bar{e}_{55}D) & 0\\ 0 & 0 & S_{66}\end{bmatrix}\begin{Bmatrix}\sigma_4\\ \sigma_5\\ \sigma_6\end{Bmatrix}\right) \tag{17}$$

where D is the scalar damage variable defined as the ratio of the effective damaged volume to the initially virgin volume of MVC. That is,

$$D=\left(\frac{a}{R}\right)^3 \tag{18}$$

The same definition of damage variable was introduced in LDT for three-dimensional solids.

Effective elastic compliances are now obtained from Eqs. (3), (4) and (17) as follows:

$$\bar{S}_{33}=\frac{1}{E}(1+\bar{e}_{33}D),\quad \bar{S}_{44}=\frac{1}{G}(1+\bar{e}_{44}D) \tag{19}$$

$\bar{S}_{55}=\frac{1}{G}(1+\bar{e}_{55}D)$, other $\bar{S}_{ij}=S_{ij}$

Equation (19) implies that damaged three-dimensional MVC of isotropic solids show in general orthotropic material macro-behavior. For the case of penny-shaped cracks ($k'=1$), it can be shown that $\bar{S}_{44}=\bar{S}_{55}$ and $\bar{S}_{66}=2(\bar{S}_{11}-\bar{S}_{12})$, which implies transversely isotropic material macro-behavior. Further discussions on macro-behavior can be found in Lee *at al.* (1997).

3. Effective Engineering Constants

In the preceding section, it is observed that damaged MVC of initially virgin isotropic solid shows orthotropic material behavior in general. Thus, the effective elastic compliances of ECM may be expressed in terms of orthotropic engineering constants. They are given for plane problems as

$$\bar{S}_{11}=\frac{1}{E_1},\quad \bar{S}_{22}=\frac{1}{E_2},$$
$$\bar{S}_{12}=-\frac{\nu_{12}}{E_1},\quad \bar{S}_{66}=\frac{1}{G_{12}} \tag{20}$$

(Lekhnitskii, 1963) and, for three-dimensional case, they are

$$\begin{aligned}
&\bar{S}_{11}=\frac{1}{E_1},\quad \bar{S}_{22}=\frac{1}{E_2},\quad \bar{S}_{33}=\frac{1}{E_3}\\
&\bar{S}_{44}=\frac{1}{G_{23}},\quad \bar{S}_{55}=\frac{1}{G_{31}},\quad \bar{S}_{66}=\frac{1}{G_{12}}\\
&\bar{S}_{12}=-\frac{\nu_{12}}{E_2},\quad \bar{S}_{21}=-\frac{\nu_{21}}{E_1},\quad \bar{S}_{13}=-\frac{\nu_{13}}{E_3}\\
&\bar{S}_{31}=-\frac{\nu_{31}}{E_1},\quad \bar{S}_{23}=-\frac{\nu_{23}}{E_3},\quad \bar{S}_{32}=-\frac{\nu_{32}}{E_2}
\end{aligned} \tag{21}$$

The effective engineering constants (i. e., effective Young's moduli E_i, effective shear moduli G_{ij}, and effective Poisson's ratios ν_{ij}) can be readily obtained from Eqs. (12) and (20) for generalized plane stress, and from Eqs. (19) and (21) for three-dimensional case. For the case of generalized plane stress, they are obtained as

$$E_1=\frac{E}{1+\bar{e}_{11}D},\quad E_2=\frac{E}{1+\bar{e}_{22}D},\quad G_{12}=\frac{G}{1+\bar{e}_{66}D} \tag{22}$$
$$\nu_{12}=\nu\left(\frac{1+\bar{e}_{12}D}{1+\bar{e}_{11}D}\right),\quad \nu_{21}=\nu\left(\frac{1+\bar{e}_{12}D}{1+\bar{e}_{22}D}\right)$$

The effective engineering constants for plane strain can be obtained from Eq. (22) by simply replacing E and ν with $E/(1-\nu^2)$ and $\nu/(1-\nu)$, respectively. In general, the elastic moduli are shown to be reduced in magnitude due to the damage, but this does not seem clear for Poisson's ratio. For line-through cracks (*i. e.*, $k'=0$), $\bar{e}_{11}$ and $\bar{e}_{12}$ vanishes so that only E_2, G_{12}, and ν_{21} are degraded. In this case the effective continuum may be weakest in the direction "2", which is normal to the crack line because E_2 is always smaller than E_1.

Similarly, the effective engineering constants for the three-dimensional case are obtained as follows:

$$\begin{aligned}
&E_1=E_2=E,\quad E_3=\frac{E}{1+\bar{e}_{33}D}\\
&G_{23}=\frac{G}{1+\bar{e}_{44}D},\quad G_{13}=\frac{G}{1+\bar{e}_{55}D},\quad G_{12}=G\\
&\nu_{12}=\nu_{21}=\nu_{31}=\nu_{32}=\nu,\quad \nu_{13}=\nu_{23}=\frac{\nu}{1+\bar{e}_{33}D}
\end{aligned} \tag{23}$$

Since E_3 is always smaller than other Young's moduli, the effective continuum may be weakest in the direction "3", which is normal to the crack surface.

4. Conclusions

On the basis of both the strain energy equivalence principle and the equivalent elliptical crack representation of local damage, the effective elastic compliances and effective engineering constants are formulated in closed form for damaged two- and three-dimensional isotropic solids. It is found that the effective Young's modulus in the direction normal to the crack surfaces is always reduced in magnitude as long as damage exists. Since the definition of damage variables defined in this study is identical to that defined in the author's previous works, the damage evolution equation as well as the damage identification methodology introduced in his previous work can be used in the damage analysis.

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